Research article

Effectiveness of lock down to curtail the spread of corona virus: A mathematical model☆

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ABSTRACT

In this paper, we have considered a mathematical model that deals with the effectiveness of the measures that may be helpful for reducing the spread of the COVID-19 virus in the society. Here we have illustrated the importance of lock down in controlling and maintaining the spread of the COVID-19 virus. The impact of the virus on the susceptible population has been considered in the model. Also, we have taken into account the susceptible population, which by taking preventive measures viz., by having strong immunity, maintaining social distancing, wearing PPE kits and masks etc., is able to reduce the possibility of getting infected from the virus. Local as well as global stability of the equilibrium points of the model have been studied using Lyapunov function and the geometrical approach techniques. Basic reproduction number has also been obtained by using the next generation matrix. To show the effectiveness of the model, different cases obtained by varying the parameters involved in the model have been considered. A comparison between the actual number of infected cases in India and that obtained by the proposed model, showing the effectiveness of the proposed model, has also been carried out.

1. Introduction

Corona virus disease (COVID-19) [1] is an infectious disease caused by a newly discovered corona virus. The virus that causes COVID-19 is mainly transmitted through droplets generated when an infected person coughs, sneezes, or exhales. These droplets are too heavy to hang in the air and quickly fall on floors or surfaces. One can get infected by breathing in the virus if he is within close proximity of someone who has COVID-19 or by touching a contaminated surface. People who are infected with COVID-19 disease experience mild, moderate and sometimes no symptoms.

Even though there is no specific treatment for the disease caused by this virus, the doctors, nurses and paramedical staff are putting their best effort to cure and save life of the patient. In doing so they are in a very close proximity with the infected patient and hence are most vulnerable of getting infected. But by wearing PPE kit, masks, gloves etc. and taking proper care they try to protect themselves from the infection. But still, many medical staff are getting infected either because the viral load in the patient is very high or due to their carelessness or not using safety kits.

As no proper medical treatment is known, till date, for curing the ailments caused by corona virus as such containing the spread of COVID-19 virus is the only possible remedy at present. Containing the COVID-19 outbreak [2,3] is a monumental challenge as the cycle of this virus was found to be around 14 to 28 days. As the symptoms of the infection appear very late, it is not possible for anyone to identify if the person to whom he is meeting is COVID-19 positive or not. Because this viral infection disease spreads through transmission its spread may be contained by improving the immunity of susceptible population, using proper protecting kits like masks, PPE kits gloves etc. and above all following the social distancing among all the population. But, as every living being has to go out for his earnings the social distancing becomes difficult to follow, whereas the immunity varies from person to person.

As such the only way to minimize the spread of the disease is not only avoiding the interaction of the uninfected person with the infected persons and the persons to whom they have met but also stopping the susceptible persons to visit the places where the infected persons have visited. This is possible only when the movement of the uninfected as well as the population which has somehow come in contact (directly or indirectly) with the infected person is restricted.

To reduce the spread of disease government of many countries took a hard decision to lockdown their countries. Even though lock down is not an easy step for any government as it serves a major disruption of the economy, still no nation was in the position to take the risk of allowing the epidemic to spread and then treating a huge number of patients. The information regarding the possible ways of protection against the disease is propagated through TV, radio, social media etc. As a result, some of the aware people use precautionary measures for their protection.

A lot of mathematical models dealing with different aspects of COVID-19 have come up recently viz., [4–8] and references therein. Most of them have considered the interaction of infected population with the susceptible population as a major cause of spread of disease. But, this disease does not only spreads through direct contact between the susceptible to the infected individuals, whereas, these infected individuals act as carriers of the virus and may spread the disease. As such, we have assumed that the disease spreads among the susceptible individuals through transmission of virus that is present not only in the infected individuals but also in the environment.

In this paper, we have considered a mathematical model that deals with the importance of lock down [6] in controlling and maintaining the spread of the COVID-19 virus. We have considered the impact of the virus on the susceptible population instead of the interaction of infected population with the susceptible population. This results in Herds transmission instead of one to one transmission, which is the case in the spread of COVID-19 virus. Also, we have taken into account the fact that a portion of the susceptible population have a lesser possibility of getting infected from the virus due to various reasons like strong immunity, taking proper measures of protection like maintaining social distancing, wearing PPE kits, masks etc.

Taking the preventive measure against the pandemic situation created worldwide due to COVID-19, government of India announced a complete lock down from 25th March, 2020 to protect its citizens from getting infected from the deadly virus. It not only affected the life of common people but also the economic activities of the country very badly. So as to overcome the economic and social problems, the lock down was relaxed in different phases starting from 14th April, 2020 and has not been removed completely yet. With the help of this mathematical model, we are interested in studying the impact of virus during the different phases of the lock down in India. We find the error between the number of predicted infected cases with the help of the proposed model and the actual infected cases as per the data provided by Indian Council of Medical Research (ICMR) during the lock down period in India. Further, as this error is under tolerable limits we are able to predict the growth of the virus and when it would be reaching its peak.

Local as well as global stability of the equilibrium points of the model have been studied using Lyapunov function and the geometrical approach techniques. Basic reproduction number, indicating the spread of the disease, has also been obtained by using the next generation matrix. The parameters of the model are firstly approximated by using the best fit least squares approximation technique. Secondly, we have considered different cases by varying the parameters and in each case obtained the reproduction number (R_0) showing whether the spread of virus is controlled or not. It turns out that a few parameters play a vital role to maintain the effectiveness of the lock down. Also, a comparative study, between the actual number of infected cases in India as reported by ICMR [9] and that obtained by the proposed model, has been carried out. Both the values are comparable. Lastly, with the help of the proposed model, we have obtained a figure predicting the total number of infected cases, number of days for the virus to spread, reach to its peak and its eradication, in India.

2. Mathematical model

We consider that, in the region under consideration the total population is $N(t)$ at time t. The total population is divided into three classes; the susceptible population $S(t)$, the infective population I(t) and recovered population $R(t)$. Also $v(t)$ be free virus present in infected person at any time t. Here it is assumed that the disease COVID-19 spreads through transmission of virus [10] to susceptible individuals by infected person or through virus spread by the infected person by coughing, sneezing, exhaling etc. at the places where he has visited. Growth rate of virus is assumed to be proportional to the number of infected individuals. The immigrants coming from different nations or places where the virus has already spread are more eligible virus carriers. They, even though are infected, stay with susceptible class of population knowingly or unknowingly. Also, a lot of susceptible individuals, to whom these infected individuals contact to, become infected and the chain continues due to which the number of infected individual grow exponentially.

The above discussed model can be represented by the following system of non-linear ordinary differential equations:-

$$\begin{cases} \frac{dS}{dt} = (1-\mu)A - \beta(1-n_p)S(1+aS)v + aI - \delta S \\[2mm] \frac{dI}{dt} = \mu A + \beta(1-n_p)S(1+aS)v - (a+\delta+\alpha+\gamma)I \\[2mm] \frac{dR}{dt} = \gamma I - \delta R \\[2mm] \frac{dv}{dt} = ac_1 I - c_2 v + c_3 v \end{cases} \quad (2.1)$$

where $S(0) \geq 0$, $I(0) \geq 0$, $R(0) \geq 0$, $v(0) \geq 0$. In the above model (2.1), let A be the number of immigrated people in the susceptible class and μ be the rate at which the immigrated population in susceptible class becomes infected, β be the rate at which a person in susceptible class moves to infected class, $n_p(0 < n_p < 1)$ is the rate at which susceptible class protect themselves from moving to the infected class by taking proper precautions like wearing protective PPE kit or mask, maintaining social distancing, improving their immunity etc., a is the rate at which the infected persons move together with the susceptible persons without taking any preventive measures like putting a mask or maintaining the social distancing. In other words, we can say that a is the measure of how much the lock down has been followed by the public. Lower the value of a higher is the effectiveness of lock down and $a = 0$ imply complete lock-down strictly followed by the public. δ is the natural rate of death, α is the disease induced death rate, γ is the recovery rate, ac_1 is the self growth rate of the virus due to increase in the infected individuals, c_2 is the natural death rate of the virus and c_3 is its growth rate, for the feasibility [11] of the model it is assumed that $c_2 - c_3 > 0$.

Using the fact that $N = S + I + R$, the above system reduces to the following system,

$$\begin{cases} \frac{dI}{dt} = \mu A + \beta(1-n_p)(N-I-R)\left(1+a(N-I-R)\right)v \\ \qquad -(a+\delta+\alpha+\gamma)I \\[2mm] \frac{dR}{dt} = \gamma I - \delta R \\[2mm] \frac{dN}{dt} = A - \delta(S+I+R) - \alpha I = A - \delta N - \alpha I \\[2mm] \frac{dv}{dt} = ac_1 I - c_2 v + c_3 v. \end{cases} \quad (2.2)$$

3. Positivity of solutions and boundedness

It is important that the model system (2.1) is epidemiologically feasible [12,13]. We will show that all the variables involved in model system (2.1) are non-negative with positive initial conditions for all time $t > 0$ as proved in the following theorem:

Theorem 1. *The solution $S(t)$, $I(t)$, $R(t)$ and $v(t)$ of model system* (2.1) *with initial conditions $S(0) \geq 0$, $I(0) \geq 0$, $R(0) \geq 0$ and $v(0) \geq 0$ are positive for all $t > 0$.*

Proof. We will show that all the variables of model (2.1) are non-negative for all time t. It follows from the second equation of (2.1) that

$$\frac{dI}{dt} \geq \mu A - (a + \delta + \alpha + \gamma)I,$$

the above equation can be written as

$$\frac{d}{dt}\left[I \exp((a + \delta + \alpha + \gamma)t)\right] \geq \mu A \exp((a + \delta + \alpha + \gamma)t).$$

Hence

$$I(t)\exp(a + \delta + \alpha + \gamma)t - I(0) \geq \int_0^t (\mu A \exp(a + \delta + \alpha + \gamma)s)\,ds,$$

$$I(t) \geq I(0)\exp(-(a + \delta + \alpha + \gamma)t)$$
$$+ \exp(-(a + \delta + \alpha + \gamma)t)\int_0^t (\mu A \exp(a + \delta + \alpha + \gamma)s)\,ds.$$

This shows that $I(t) > 0$ for all $t > 0$. Further, from the first equation of (2.1), we have

$$\frac{dS}{dt} = (1 - \mu)A + aI(t) - \left(\beta(1 - n_p)(1 + aS(t))v(t) + \delta\right)S(t),$$

thus,

$$\frac{dS}{dt} + f_1(s)S(t) = (1 - \mu)A + aI(t),$$

where $f_1(s) = \left(\beta(1 - n_p)(1 + aS(t))v(t) + \delta\right)$. This implies that

$$\frac{d}{dt}\left(S(t)\exp\left(\int_0^t f_1(s)ds\right)\right)$$
$$= ((1 - \mu)A + aI(t))\exp\left(\int_0^t f_1(s)ds\right),$$

hence we obtain that

$$S(t) = \exp\left(-\int_0^t f_1(s)ds\right)$$
$$\times \left[S(0) + \int_0^t ((1 - \mu)A + aI(s))\exp\left(\int_0^s f_1(x)dx\right)ds\right].$$

This shows that $S(t) > 0$ for all $t > 0$. Similarly, it is easy to show that $R(t) > 0$ and $v(t) > 0$ for all $t > 0$. Thus, the solution $S(t)$, $I(t)$, $R(t)$ and $v(t)$ of model system (2.1) with initial conditions $S(0) > 0$, $I(0) \geq 0$, $R(0) \geq 0$ and $v(0) \geq 0$ are positive for all $t > 0$. Hence the proof. $\square$

Theorem 2. *All the solutions of the model* (2.2) *are bounded.*

Proof. The system (2.2) is split into two parts, the human population (*i.e.*, $S(t)$, $I(t)$, $R(t)$) and free virus (*i.e.*, $v(t)$). It follows from the first three equations of the system (2.2) that

$$\frac{dN}{dt} \leq A - \delta N(t).$$

Multiply the integrating factor $e^{\delta t}$ of the above equation and integrating from 0 to t, we obtain

$$N(t) \leq N_0 e^{-\delta t} + \frac{A}{\delta}. \tag{3.1}$$

It implies that, $\lim_{t \to \infty} \sup N(t) \leq \frac{A}{\delta}$. Thus, $0 \leq N(t) \leq \frac{A}{\delta}$ for all $t > 0$. Since $N(t) = S(t) + I(t) + R(t)$,

$$0 \leq I(t) + R(t) \leq N(t) \leq \frac{A}{\delta}.$$

From the fourth equation of the model (2.2), and using the fact that $I(t) \leq \frac{A}{\delta}$ for all $t > 0$, we have

$$\frac{dv}{dt} \leq \frac{ac_1 A}{\delta} - (c_2 - c_3)v(t).$$

Proceeding in a similar way as done for (3.1), we obtain $\lim_{t \to \infty} \sup v(t) \leq \frac{ac_1 A}{\delta(c_2 - c_3)}$. This implies that $0 \leq v(t) \leq \frac{ac_1 A}{\delta(c_2 - c_3)}$ for all $t > 0$.

From the above discussion, we can see the feasible region of the human population for system (2.2) is

$$\Omega_H = \left\{(I, R, N): \ 0 \leq I + R \leq N \leq \frac{A}{\delta}\right\}$$

and feasible region of virus population for system (2.2) is

$$\Omega_v = \left\{v: \ 0 \leq v \leq \frac{ac_1 A}{\delta(c_2 - c_3)}\right\}$$

which proves the theorem. $\square$

Define $\Omega = \Omega_H \times \Omega_v$. Now, Ω is a positive invariant for the system (2.2). Let $Int(\Omega)$ denote the interior of Ω.

4. Equilibrium analysis

The above system (2.2) have three non-negative equilibrium points as follows:

4.1. Disease free equilibrium point $E_0(I_0, R_0, N_0, v_0)$

This equilibrium point is obtained by equating to zero the right hand side of the system of Eq. (2.2) together with $I_0 = 0$, $R_0 = 0$ and $v_0 = 0$, which gives $N_0 = \frac{A}{\delta}$.

4.1.1. Basic reproduction number

Here, we obtain the basic reproduction number of model system (2.2), at E_0, using next generation matrix approach [14,15]. For our model (2.2) $\mathcal{F}$ and $\mathcal{V}$ are given as:

$$\mathcal{F} = \begin{bmatrix} \mu A + \beta(1 - n_p)(N - I - R)(1 + a(N - I - R))v \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\mathcal{V} = \begin{bmatrix} (a + \delta + \alpha + \gamma)I \\ -\gamma I + \delta R \\ -A + \delta N + \alpha I \\ -ac_1 I + (c_2 - c_3)v \end{bmatrix}.$$

Let F and G be the Jacobian's of $\mathcal{F}$ and $\mathcal{V}$ respectively at disease free equilibrium point which are given as:

$$F = \begin{bmatrix} 0 & \beta(1 - n_p)N_0(1 + aN_0) \\ 0 & 0 \end{bmatrix},$$

and

$$G = \begin{bmatrix} (a + \delta + \alpha + \gamma) & 0 \\ -ac_1 & (c_2 - c_3) \end{bmatrix}$$

with

$$G^{-1} = \frac{1}{(c_2 - c_3)(a + \delta + \alpha + \gamma)}$$
$$\times \begin{bmatrix} (a + \delta + \alpha + \gamma) & 0 \\ ac_1 & (c_2 - c_3) \end{bmatrix}.$$

Now the basic reproduction number R_0 [8] of the model is the spectral radius of the matrix FG^{-1}, which is

$$\mathcal{R}_0 = \frac{ac_1 \beta(1 - n_p)N_0(1 + aN_0)}{(c_2 - c_3)(a + \delta + \alpha + \gamma)} = \frac{aAc_1 \beta(1 - n_p)(\delta + aA)}{\delta^2(c_2 - c_3)(a + \delta + \alpha + \gamma)}.$$

Special Case: Since an increment in the recovered population $R(t)$ depends only on recovery rate γ, thus, in particular, if $\gamma = 0$, then

$$\mathcal{R}_{01} = \frac{aAc_1\beta(1 - n_p)(\delta + aA)}{\delta^2(c_2 - c_3)(a + \delta + \alpha)}.$$

It is easy to observe from the above cases and expressions that $\mathcal{R}_{01}$ is bigger than $\mathcal{R}_0$.

4.2. Non-removal class equilibrium point $E_1(I^\#, R^\#, N^\#, v^\#)$

Equating to zero the right hand side of the system of Eq. (2.2) together with $R^\# = 0$, which gives

$$\mu A + \beta(1 - n_p)(N^\# - I^\#)\left(1 + a(N^\# - I^\#)\right)v^\#$$
$$- (a + \delta + \alpha)I^\# = 0 \qquad (4.1)$$
$$A - \delta N^\# - \alpha I^\# = 0 \qquad (4.2)$$
$$ac_1I^\# - c_2v^\# + c_3v^\# = 0 \qquad (4.3)$$

From Eqs. (4.2) and (4.3),

$$N^\# = \frac{A - \alpha I^\#}{\delta}, \text{ where}(A - \alpha^\#) > 0$$

$$v^\# = \frac{ac_1I^\#}{(c_2 - c_3)}, \text{ where}(c_2 - c_3) > 0$$

Putting the above value $N^\#$ and $v^\#$ in (4.1), we get

$$b_1(I^\#)^3 - b_2(I^\#)^2 + b_3I^\# + b_4 = 0$$

where $b_1 = \beta(1-n_p)a^2c_1(\alpha+\delta)^2$, $b_2 = \beta(1-n_p)ac_1(\alpha+\delta)(2aA+\delta)$, $b_3 = (a + \delta + \alpha)(c_2 - c_3)\delta^2(\mathcal{R}_{01} - 1)$ and $b_4 = \mu A(c_2 - c_3)\delta^2$.

If $\mathcal{R}_{01} > 1$, then $b_1, b_2, b_4 > 0$ and $b_3 > 0$. By Descartes' rule of signs at least one positive root must exist.

4.3. Endemic equilibrium point $E_2(I^*, R^*, N^*, v^*)$

Equating to zero the right hand side of the system of Eq. (2.2), which gives

$$\mu A + \beta(1 - n_p)(N^* - I^* - R^*)\left(1 + a(N^* - I^* - R^*)\right)v^*$$
$$- (a + \delta + \alpha + \gamma)I^* = 0, \qquad (4.4)$$
$$\gamma I^* - \delta R^* = 0, \qquad (4.5)$$
$$A - \delta N^* - \alpha I^* = 0 \qquad (4.6)$$

and

$$ac_1I^* - c_2v^* + c_3v^* = 0. \qquad (4.7)$$

From Eq. (4.5), (4.6) and (4.7), we get

$$R^* = \frac{\gamma I^*}{\delta}$$

$$N^* = \frac{A - \alpha I^*}{\delta}, \text{ where } (A - \alpha I^*) > 0$$

$$v^* = \frac{ac_1I^*}{(c_2 - c_3)}, \text{ where } (c_2 - c_3) > 0.$$

Putting the above value R^*, N^* and v^* in Eq. (4.4), we get

$$m_1(I^*)^3 - m_2(I^*)^2 + m_3I^* + m_4 = 0$$

where $m_1 = \beta(1-n_p)a^2c_1(\alpha + \gamma + \delta)^2$, $m_2 = \beta(1-n_p)ac_1(\alpha + \gamma + \delta)(2aA + \delta)$, $m_3 = (a + \delta + \alpha + \gamma)(c_2 - c_3)\delta^2(\mathcal{R}_0 - 1)$ and $m_4 = \mu A(c_2 - c_3)\delta^2$.

If $\mathcal{R}_0 > 1$, then $b_1, b_2, b_4 > 0$ and $b_3 > 0$. By Descartes' rule of signs at least one positive sign must exist.

5. Stability analysis of the equilibrium points

In this section we discuss the stability of the model at the three equilibrium points obtained in the preceding section.

5.1. Stability analysis of the disease free equilibrium

In this section, we obtain the local and global stability conditions for disease free equilibrium point E_0 given in section 4.1.

Theorem 3. *The disease free equilibrium point E_0, given in section 4.1, is locally asymptotically stable for $\mathcal{R}_0 \leq 1$.*

Proof. The variational matrix $V(E_0)$ around equilibrium point $E_0(I_0, R_0, N_0, v_0)$ is given by

$$V(E_0) = \begin{bmatrix} -(a + \delta + \alpha + \gamma) & 0 & 0 & \beta(1 - n_p)(\delta + aA)\frac{A}{\delta^2} \\ \gamma & -\delta & 0 & 0 \\ -\alpha & 0 & -\delta & 0 \\ ac_1 & 0 & 0 & -(c_2 - c_3) \end{bmatrix}.$$

Its characteristic equation is

$$(\delta + \lambda)(\delta + \lambda)(\lambda^2 + P_1\lambda + P_0) = 0, \qquad (5.1)$$

where

$$P_1 = (a + \delta + \alpha + \gamma) + (c_2 - c_3),$$
$$P_0 = (a + \delta + \alpha + \gamma)(c_2 - c_3)(1 - \mathcal{R}_0).$$

Clearly there are repeated roots at $\lambda = -\delta < 0$. Now by Routh–Hurwitz criterion, the other two roots of the characteristic equation (5.1) will be negative if both P_0 and P_1 are positive. Since $c_2 > c_3$ thus $P_0 > 0$ and $P_1 > 0$ when $\mathcal{R}_0 < 1$.

Thus we conclude that the disease free equilibrium point E_0 of the model system (2.2) is locally asymptotically stable if $\mathcal{R}_0 \leq 1$. □

Theorem 4. *The disease-free equilibrium E_0 is globally asymptotically stable for $\mathcal{R}_0 \leq 1$.*

Proof. Consider the following Lyapunov function:

$$L(t) = ac_1I(t) + (a + \delta + \alpha + \gamma)v(t).$$

On differentiating the above equation w.r.t. t and using (2.2), we get,

$$\frac{dL}{dt} = ac_1\mu A + ac_1\beta(1 - n_p)(N - I - R)\left(1 + a(N - I - R)\right)v$$
$$- (a + \delta + \alpha + \gamma)(c_2 - c_3)v$$

therefore,

$$L' \leq ac_1\beta(1 - n_p)\frac{A}{\delta}\left(1 + \frac{aA}{\delta}\right)v - (a + \delta + \alpha + \gamma)(c_2 - c_3)v$$

$$L' \leq (a + \delta + \alpha + \gamma)(c_2 - c_3)\left(\frac{aAc_1\beta(1 - n_p)(\delta + aA)}{\delta^2(c_2 - c_3)(a + \delta + \alpha + \gamma)} - 1\right)v$$

$$L' \leq (a + \delta + \alpha + \gamma)(c_2 - c_3)(\mathcal{R}_0 - 1)v$$

Hence, $L' \leq 0$ if $\mathcal{R}_0 \leq 1$. By Lyapunov–Lasalle's Theorem [16], we observe that the system (2.2) has the maximum invariant set for $L' = 0$ if and only if $\mathcal{R}_0 \leq 1$ and $I = v = 0$. □

5.1.1. Persistence

The epidemiological implication of Theorem 4 is that the infected fraction (i.e., I and v) goes to zero in time when $\mathcal{R}_0 \leq 1$, that is, the COVID-19 eventually disappears from the population. The disease is endemic if the infected fraction remains above a

certain positive level for sufficiently large time [17]. System (2.2) can be defined to be uniformly persistent if

$$\min\{\lim_{t\to\infty}\inf I(t), \lim_{t\to\infty}\inf R(t), \lim_{t\to\infty}\inf N(t), \lim_{t\to\infty}\inf v(t)\} > \epsilon$$

for some $\epsilon > 0$ for all initial points in $int(\Omega)$.

Theorem 5. *The model system (2.2) is uniformly persistent in $int(\Omega)$ if $\mathcal{R}_0 > 1$.*

Proof. Ω is positively invariant with respect to the model. The disease free equilibrium is unique and located on the boundary of Ω. On using the result stated in theorem (4.3) in [18], the unstability of the disease free equilibrium implies the uniform persistence of the model system (2.2). □

5.2. Stability analysis of non-removal class equilibrium point

In this section, first, we obtain the condition for the local stability of the endemic equilibrium point E_1 using the Routh–Hurwitz criterion followed by global stability of the point using the Lyapunov's method [19,20].

5.2.1. Local stability of non-removal class equilibrium point

Theorem 6. *If $\mathcal{R}_{01} > 1$ then the endemic equilibrium point is locally asymptotically stable provided $\rho_i > 0$, where $i = 1, 2, 3$ and $\rho_1\rho_2 - \rho_3 > 0$.*

Proof. The variational matrix $V(E_1)$ around equilibrium point $E_1(I^{\#}, R^{\#}, N^{\#}, v^{\#})$ is given by

$$V(E_1) = \begin{bmatrix} -b_{11} & -b_{12} & -b_{13} & b_{14} \\ \gamma & -\delta & 0 & 0 \\ -\alpha & 0 & -\delta & 0 \\ ac_1 & 0 & 0 & -(c_2 - c_3) \end{bmatrix}.$$

where

$$b_{11} = \left(\beta(1 - n_p)(1 + 2a(N^{\#} - I^{\#}))v^+ + (a + \delta + \alpha + \gamma)\right),$$
$$b_{12} = \beta(1 - n_p)\left(1 + 2a(N^{\#} - I^{\#})\right)v^{\#},$$
$$b_{13} = \beta(1 - n_p)\left(1 + 2a(N^{\#} - I^{\#})\right)v^{\#}$$

and

$$b_{14} = \beta(1 - n_p)\left((N^{\#} - I^{\#}) + a(N^{\#} - I^{\#})^2\right).$$

Therefore the corresponding characteristic equation is

$$(\lambda + \delta)(\lambda^3 + \rho_1\lambda^2 + \rho_2\lambda + \rho_3) = 0,$$

where λ is the eigenvalue with the first eigenvalue $\lambda = -\delta$ and the other eigenvalues are given by the equation

$$\lambda^3 + \rho_1\lambda^2 + \rho_2\lambda + \rho_3 = 0, \tag{5.2}$$

where

$$\rho_1 = \delta - a_{11} + c_2 - c_3$$
$$\rho_2 = -\delta a_{11} - \gamma a_{12} + \alpha a_{13} - aa_{14}c_1 + \delta c_2 - a_{11}c_2 - \delta c_3 + a_{11}c_3,$$
$$\rho_3 = -a\delta a_{14}c_1 - \delta a_{11}c_2 - \gamma a_{12}c_2 + \alpha a_{13}c_2$$
$$\quad + \delta a_{11}c_3 + \gamma a_{12}c_3 - \alpha a_{13}c_3.$$

Since ρ_1, ρ_2 and ρ_3 are positive and by algebraic calculations we have that $\rho_1\rho_2 - \rho_3 > 0$, thus by Routh–Hurwitz criterion all roots of (5.2) are either negative or have negative real parts. Therefore equilibrium point is locally asymptotically stable. □

5.2.2. Global stability of non-removal class equilibrium point

Theorem 7. *The equilibrium point $E_1(I^{\#}, 0, N^{\#}, M^{\#})$ is globally asymptotically stable if the following inequalities hold true:*

$$\left(\beta(1 - n_p)(1 + 2a(N^{\#} - I^{\#}))v^{\#}\right)(\alpha - \delta)^2 <$$
$$6k\delta^2\left(\beta(1 - n_p)(1 + 2a(N^{\#} - I^{\#}))v^{\#} + (a + \delta + \alpha)\right),$$

and

$$(ac_1)\left(\beta(1 - n_p)((N^{\#} - I^{\#}) + a(N^{\#} - I^{\#})^2)\right.$$
$$\left. + (a + \delta + \alpha)(c_2 - c_3)\right)^2 <$$
$$3k(a + \delta + \alpha)(c_2 - c_3)^2\left(\beta(1 - n_p)(1 + 2a(N^{\#} - I^{\#}))v^{\#}\right.$$
$$\left. + (a + \delta + \alpha)\right).$$

Proof. Consider a positive definite function U such that

$$U = \frac{1}{2}i^2 + \frac{p_1}{2}n^2 + \frac{p_2}{2}m^2$$

where p_1 and p_2 are positive constants and i, n and v are small perturbation from the equilibrium point E_2, that is, $I = I^{\#} + i$, $N = N^{\#} + n$ and $v = v^{\#} + m$. On differentiating U with respect to t, we get

$$\frac{dU}{dt} = i\frac{dI}{dt} + p_1n\frac{dN}{dt} + p_2m\frac{dv}{dt}$$

which due to (2.2), gives

$$\frac{dU}{dt} = i[\mu A + \beta(1 - n_p)(N - I)(1 + a(N - I))v - (a + \delta + \alpha)I]$$
$$+ p_1n[A - \delta N - \alpha I] + p_2m[ac_1I - c_2v + c_3v]$$
$$= i[\mu A + \beta(1 - n_p)(N^{\#} + n - I^{\#} - i)$$
$$\times\left(1 + a(N^{\#} + n - I^{\#} - i)\right)(v^{\#} + m)$$
$$- (a + \delta + \alpha)(I^{\#} + i)] + p_1n[A - \delta(N^{\#} + n) - \alpha(I^{\#} + i)]$$
$$+ p_2m[ac_1(I^{\#} + i) - (c_2 - c_3)(v^{\#} + m)]$$
$$= i[\beta(1 - n_p)\left((n - i)v^{\#} + (N^{\#} - I^{\#})m\right.$$
$$+ a(N^{\#} - I^{\#})^2m + 2a(N^{\#} - I^{\#})v^{\#}(n - i))$$
$$- (a + \delta + \alpha)i] + p_1n[-\delta n - \alpha i] + p_2m[ac_1i - (c_2 - c_3)m]$$
$$= \beta(1 - n_p)((N^{\#} - I^{\#}) + a(N^{\#} - I^{\#})^2)im$$
$$+ \beta(1 - n_p)(1 + 2a(N^{\#} - I^{\#}))v^{\#}(in - i^2)$$
$$- (a + \delta + \alpha)i^2 - p_1\delta n^2 - \alpha p_1in + p_2ac_1im$$
$$- p_2(c_2 - c_3)m^2$$
$$= -\left(\beta(1 - n_p)(1 + 2a(N^{\#} - I^{\#}))v^{\#} + (a + \delta + \alpha)\right)i^2$$
$$+ \left(\beta(1 - n_p)((N^{\#} - I^{\#}) + a(N^{\#} - I^{\#})^2) + p_2ac_1\right)im$$
$$+ \left(\beta(1 - n_p)(1 + 2a(N^{\#} - I^{\#}))v^{\#} - \alpha p_1\right)in$$
$$- p_1\delta n^2 - p_2(c_2 - c_3)m^2.$$

$\frac{dU}{dt}$ will be negative definite provided, for $k > 1$, the inequalities

$$\left(\beta(1 - n_p)(1 + 2a(N^{\#} - I^{\#}))v^{\#} - \alpha p_1\right)^2 <$$
$$6kp_1\delta\left(\beta(1 - n_p)(1 + 2a(N^{\#} - I^{\#}))v^{\#} + (a + \delta + \alpha)\right),$$

and

$$\left(\beta(1 - n_p)((N^{\#} - I^{\#}) + a(N^{\#} - I^{\#})^2) + p_2ac_1\right)^2 <$$
$$3kp_2(c_2 - c_3)\left(\beta(1 - n_p)(1 + 2a(N^{\#} - I^{\#}))v^{\#} + (a + \delta + \alpha)\right)$$

hold. On choosing

$$p_1 = \frac{\delta\beta(1 - n_p)(1 + 2a(N^{\#} - I^{\#}))v^{\#}}{\alpha^2}$$

and

$$p_2 = \frac{(a + \delta + \alpha)(c_2 - c_3)}{ac_1}$$

the above inequalities may be combined from which the theorem follows. □

5.3. Stability analysis of endemic equilibrium point

In this section, firstly we obtain the condition for the local stability of the endemic equilibrium point E_2 using the Routh–Hurwitz criterion followed by global stability of the point using geometric approach [21].

5.3.1. Local stability of endemic equilibrium

Theorem 8. *If $\mathcal{R}_0 > 1$ then the endemic equilibrium point is locally asymptotically stable provided $\sigma_i > 0$, where $i = 1, 2, 3$ and $\sigma_1\sigma_2 - \sigma_3 > 0$.*

Proof. The variational matrix $V(E_2)$ around equilibrium point $E_2(I^*, R^*, N^*, v^*)$ is given by

$$V(E_2) = \begin{bmatrix} -a_{11} & -a_{12} & a_{13} & a_{14} \\ \gamma & -\delta & 0 & 0 \\ -\alpha & 0 & -\delta & 0 \\ ac_1 & 0 & 0 & -(c_2 - c_3) \end{bmatrix}.$$

where

$a_{11} = \left(\beta(1 - n_p)(1 + 2a(N^* - I^* - R^*))v^* + (a + \delta + \alpha + \gamma)\right),$

$a_{12} = \beta(1 - n_p)\left(1 + 2a(N^* - I^* - R^*)\right)v^*,$

$a_{13} = \beta(1 - n_p)\left(1 + 2a(N^* - I^* - R^*)\right)v^*$

and

$a_{14} = \beta(1 - n_p)\left((N^* - I^* - R^*) + a(N^* - I^* - R^*)^2\right).$

Therefore corresponding characteristic equation is

$$(\lambda + \delta)(\lambda^3 + \sigma_1\lambda^2 + \sigma_2\lambda + \sigma_3) = 0,$$

where λ is the eigenvalue and the one of the eigenvalue is $\lambda = -\delta$, and other eigenvalues are given by

$$\lambda^3 + \sigma_1\lambda^2 + \sigma_2\lambda + \sigma_3 = 0, \tag{5.3}$$

where

$\sigma_1 = \delta - a_{11} + c_2 - c_3,$

$\sigma_2 = -\delta a_{11} - \gamma a_{12} + \alpha a_{13} - aa_{14}c_1 + \delta c_2 - a_{11}c_2 - \delta c_3 + a_{11}c_3,$

$\sigma_3 = -a\delta a_{14}c_1 - \delta a_{11}c_2 - \gamma a_{12}c_2 + \alpha a_{13}c_2 + \delta a_{11}c_3$
$\quad + \gamma a_{12}c_3 - \alpha a_{13}c_3.$

Since σ_1, σ_2 and σ_3 are positive and algebraic manipulation convey that $\sigma_1\sigma_2 - \sigma_3 > 0$, thus by Routh–Hurwitz criterion all roots of (5.3) are either negative or have negative real parts. Therefore equilibrium point is locally asymptotically stable. □

5.3.2. Global stability of endemic equilibrium $E_2(I^*, R^*, N^*, v^*)$

In the following, will apply the second compound matrix techniques and autonomous convergence theorems developed by Smith [22], LiandMuldowney [23] to demonstrate the global stability of E_2.

Theorem 9. *The unique endemic equilibrium is globally stable if $\mathcal{R}_0 > 1$ provided that the following inequality:*

$$\max\{-\mu\frac{A}{I}, (-\beta(1 - n_p)(1 + 2a(N - I - R))v - \mu\frac{A}{I}$$
$$- \delta + a + \alpha + \gamma)\} < -\mathcal{X}$$

is satisfied, where $\mathcal{X} > 0$ is a constant.

Proof. The Jacobian matrix J of model system (2.2) at an arbitrary point (I, R, N, v) is given as

$$J = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \gamma & -\delta & 0 & 0 \\ -\alpha & 0 & -\delta & 0 \\ ac_1 & 0 & 0 & -(c_2 - c_3) \end{bmatrix},$$

where

$a_{11} = -\left(\beta(1 - n_p)(1 + 2a(N - I - R))v + (a + \delta + \alpha + \gamma)\right),$

$a_{12} = -\beta(1 - n_p)\left(1 + 2a(N - I - R)\right)v,$

$a_{13} = \beta(1 - n_p)\left(1 + 2a(N - I - R)\right)v$

and

$a_{14} = \beta(1 - n_p)\left((N - I - R) + a(N - I - R)^2\right).$

The second compound matrix $J^{[2]}$ is

$$J^{[2]} = \begin{bmatrix} j_{11} & 0 & 0 & -a_{13} & -a_{14} & 0 \\ 0 & j_{22} & 0 & a_{12} & 0 & -a_{14} \\ 0 & 0 & j_{33} & 0 & a_{12} & a_{13} \\ \alpha & \gamma & 0 & j_{44} & 0 & 0 \\ -ac_1 & 0 & \gamma & 0 & j_{55} & 0 \\ 0 & ac_1 & -\alpha & 0 & 0 & j_{66} \end{bmatrix},$$

where

$j_{11} = -\left(\beta(1 - n_p)(1 + 2a(N - I - R))v + (a + 2\delta + \alpha + \gamma)\right),$

$j_{22} = -\left(\beta(1 - n_p)(1 + 2a(N - I - R))v + (a + 2\delta + \alpha + \gamma)\right),$

$j_{33} = -\left(\beta(1 - n_p)(1 + 2a(N - I - R))v + (a + \delta + \alpha + \gamma)\right.$
$\quad \left. + (c_2 - c_3)\right),$

$j_{44} = -2\delta,$

$j_{55} = -\delta - (c_2 - c_3)$

and

$j_{66} = -\delta - (c_2 - c_3).$

Let

$$Q = \begin{bmatrix} \frac{1}{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{I} & 0 & 0 \\ 0 & 0 & \frac{1}{v} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{v} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{v} \end{bmatrix}.$$

Next, using the second compound matrix $J^{[2]}$ and Q_f is the directional derivative of Q in the direction of the vector field f, we calculate the following [24,25],

$$\mathcal{B} = Q_f Q^{-1} + QJ^{[2]}Q^{-1}.$$

which is obtained as

$$\mathcal{B} = \begin{bmatrix} j_{11} - \frac{I'}{I} & 0 & -a_{13} & 0 & -a_{14}\frac{v}{I} & 0 \\ 0 & j_{22} - \frac{I'}{I} & a_{12} & 0 & 0 & -a_{14}\frac{v}{I} \\ \alpha & \gamma & j_{44} - \frac{I'}{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & j_{33} - \frac{v'}{v} & a_{12} & a_{13} \\ -ac_1\frac{I}{v} & 0 & 0 & \gamma & j_{55} - \frac{v'}{v} & 0 \\ 0 & -ac_1\frac{I}{v} & 0 & -\alpha & 0 & j_{66} - \frac{v'}{v} \end{bmatrix}.$$

Since, we have

$$\frac{I'}{I} = \mu\frac{A}{I} + \beta(1 - n_p)(N - I - R)\left(1 + a(N - I - R)\right)\frac{v}{I} - (a + \delta + \alpha + \gamma),$$

$$\frac{v'}{v} = ac_1\frac{I}{v} - (c_2 - c_3)$$

thus,

$$
\mathcal{B} =
\begin{bmatrix}
A_{11} & 0 & A_{13} & 0 & A_{15} & 0 \\
0 & A_{22} & A_{23} & 0 & 0 & A_{26} \\
\alpha & \gamma & A_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & A_{44} & A_{45} & A_{46} \\
-ac_1\frac{I}{v} & 0 & 0 & \gamma & A_{55} & 0 \\
0 & -ac_1\frac{I}{v} & 0 & -\alpha & 0 & A_{66}
\end{bmatrix},
$$

where,

$$
A_{11} = -\beta(1-n_p)(1+2a(N-I-R))v - \mu\frac{A}{I}
$$
$$
- \beta(1-n_p)(N-I-R)(1+a(N-I-R))\frac{v}{I},
$$

$$
A_{22} = -\beta(1-n_p)(1+2a(N-I-R))v - \mu\frac{A}{I}
$$
$$
- \beta(1-n_p)(N-I-R)(1+a(N-I-R))\frac{v}{I},
$$

$$
A_{33} = -\mu\frac{A}{I} - \beta(1-n_p)(N-I-R)(1+a(N-I-R))\frac{v}{I}
$$
$$
+ (a-\delta+\alpha+\gamma),
$$

$$
A_{44} = -\beta(1-n_p)(1+2a(N-I-R))v
$$
$$
- (a+\delta+\alpha+\gamma) - ac_1\frac{I}{v},
$$

$$
A_{55} = -\delta - ac_1\frac{I}{v},
$$

$$
A_{66} = -\delta - ac_1\frac{I}{v},
$$

$$
A_{13} = -\beta(1-n_p)(1+2a(N-I-R))v,
$$

$$
A_{15} = -\beta(1-n_p)\left((N-I-R)+a(N-I-R)^2\right)\frac{v}{I},
$$

$$
A_{23} = -\beta(1-n_p)(1+2a(N-I-R))v,
$$

$$
A_{26} = -\beta(1-n_p)\left((N-I-R)+a(N-I-R)^2\right)\frac{v}{I},
$$

$$
A_{45} = -\beta(1-n_p)(1+2a(N-I-R))v,
$$

and

$$
A_{46} = \beta(1-n_p)(1+2a(N-I-R))v.
$$

Now, we define the following norm on R^6

$$
\|z\| = \sup\{U_1, U_2\}
$$

where $z \in R^6$, with components $z_i(i = 1, 2, 3, 4, 5, 6)$ and

$$
U_1(z_1, z_2, z_3)
$$
$$
=
\begin{cases}
\max\{|z_1|, |z_2|+|z_3|\}, & \text{if } sgn(z_1) = sgn(z_2) = sgn(z_3), \\
\max\{|z_2|, |z_1|+|z_3|\}, & \text{if } sgn(z_1) = sgn(z_2) = -sgn(z_3), \\
\max\{|z_1|, |z_2|, |z_3|\}, & \text{if } sgn(z_1) = -sgn(z_2) = sgn(z_3), \\
\max\{|z_1|+|z_3|, |z_2|+|z_3|\}, & \text{if } -sgn(z_1) = sgn(z_2) = sgn(z_3),
\end{cases}
$$

$$
U_2(z_4, z_5, z_6)
$$
$$
=
\begin{cases}
|z_4|+|z_5|+|z_6|, & \text{if } sgn(z_4) = sgn(z_5) = sgn(z_6), \\
\max\{|z_4|+|z_5|, |z_4|+|z_6|\}, & \text{if } sgn(z_4) = sgn(z_5) = -sgn(z_6), \\
\max\{|z_5|, |z_4|+|z_6|\}, & \text{if } sgn(z_4) = -sgn(z_5) = sgn(z_6), \\
\max\{|z_4|+|z_6|, |z_5|+|z_6|\}, & \text{if } -sgn(z_4) = sgn(z_5) = sgn(z_6).
\end{cases}
$$

Case 1. $U_1(z) > U_2(z)$ **and** $z_1, z_2, z_3 > 0$
In this case,

$$
\|z\| = \max\{|z_1|, |z_2|+|z_3|\}.
$$

Subcase 1.1: $|z_1| > |z_2| + |z_3|$.
Then $\|z\| = |z_1| = z_1$ and $U_2(z) < |z_1|$. Taking the right hand derivative of $\|z\|$, we obtain

$$
D_+\|z\| = z_1' = A_{11}z_1 + A_{13}z_3 + A_{15}z_5
$$

$$
D_+\|z\| = (-\beta(1-n_p)(1+2a(N-I-R))v
$$
$$
- \beta(1-n_p)\left((N-I-R)+a(N-I-R)^2\right)\frac{v}{I}
$$
$$
- \mu\frac{A}{I})z_1 + (-\beta(1-n_p)(1+2a(N-I-R))v)z_3
$$
$$
+ (-\beta(1-n_p)\left((N-I-R)+a(N-I-R)^2\right)\frac{v}{I})z_5
$$
$$
\leq (-\beta(1-n_p)(1+2a(N-I-R))v
$$
$$
- \beta(1-n_p)\left((N-I-R)+a(N-I-R)^2\right)\frac{v}{I}
$$
$$
- \mu\frac{A}{I})|z_1| + (\beta(1-n_p)(1+2a(N-I-R))v)|z_3|
$$
$$
+ (\beta(1-n_p)\left((N-I-R)+a(N-I-R)^2\right)\frac{v}{I})|z_5|.
$$

Since $|z_3| < |z_1|$ and $|z_5| \leq U_2(z) < |z_1|$, we have

$$
D_+\|z\| \leq -\frac{\mu A}{I}\|z\|. \tag{5.4}
$$

Subcase 1.2: $|z_1| < |z_2| + |z_3|$.
Then $\|z\| = |z_2| + |z_3| = z_2 + z_3$ and $U_2(z) < |z_2| + |z_3|$. On taking the right hand derivative of $\|z\|$, we obtain

$$
D_+\|z\| = z_2' + z_3' = A_{22}z_2 + A_{23}z_3 + A_{26}z_6 + \alpha z_1 + \gamma z_2 + A_{33}z_3
$$

$$
D_+\|z\| = (-\beta(1-n_p)(1+2a(N-I-R))v
$$
$$
- \beta(1-n_p)\left((N-I-R)+a(N-I-R)^2\right)\frac{v}{I}
$$
$$
- \mu\frac{A}{I})z_2 + (-\beta(1-n_p)(1+2a(N-I-R))v)z_3
$$
$$
+ (-\beta(1-n_p)\left((N-I-R)+a(N-I-R)^2\right)\frac{v}{I})z_6
$$
$$
+ \alpha z_1 + \gamma z_2
$$
$$
+ (-\mu\frac{A}{I} - \beta(1-n_p)(N-I-R)(1+a(N-I-R))\frac{v}{I}
$$
$$
+ (a-\delta+\alpha+\gamma))z_3
$$
$$
= \alpha z_1 + (-\beta(1-n_p)(1+2a(N-I-R))v
$$
$$
- \beta(1-n_p)\left((N-I-R)+a(N-I-R)^2\right)\frac{v}{I}
$$
$$
- \mu\frac{A}{I} + \gamma)z_2 + (-\beta(1-n_p)((N-I-R)
$$
$$
+ a(N-I-R)^2)\frac{v}{I})z_6
$$
$$
+ (-\beta(1-n_p)(1+2a(N-I-R))v
$$
$$
- \beta(1-n_p)(N-I-R)(1+a(N-I-R))\frac{v}{I}
$$
$$
- \mu\frac{A}{I} + (a-\delta+\alpha+\gamma))z_3
$$
$$
\leq (-\beta(1-n_p)(1+2a(N-I-R))v
$$
$$
- \beta(1-n_p)\left((N-I-R)+a(N-I-R)^2\right)\frac{v}{I}
$$
$$
- \mu\frac{A}{I} + \gamma + \alpha)|z_2| + (\beta(1-n_p)((N-I-R)
$$
$$
+ a(N-I-R)^2)\frac{v}{I})|z_6|
$$

$$+ (-\beta(1 - n_p)(1 + 2a(N - I - R))\, v$$
$$- \beta(1 - n_p)(N - I - R)(1 + a(N - I - R))\frac{v}{I}$$
$$- \mu\frac{A}{I} + (a - \delta + 2\alpha + \gamma))|z_3|$$

Since $|z_6| < U_2(z) < |z_2| + |z_3|$, we obtain

$$D_+\|z\| \leq (-\beta(1 - n_p)(1 + 2a(N - I - R))v - \mu\frac{A}{I} - \delta + 2\alpha + a + \gamma)\|z\|.$$
$$\tag{5.5}$$

Case 2. $U_1(z) > U_2(z)$ **and** $z_1 < 0 < z_2, z_3$
In this case

$$\|z\| = \max\{|z_1| + |z_3|, |z_2| + |z_3|\}.$$

Subcase 2.1: $|z_1| > |z_2|$.
Then $\|z\| = |z_1| + |z_3| = -z_1 + z_3$ and $U_2(z) < |z_1| + |z_3|$. On taking the right hand derivative of $\|z\|$, we obtain

$$D_+\|z\| = -z_1' + z_3' = -A_{11}z_1 - A_{13}z_3 - A_{15}z_5 + \alpha z_1 + \gamma z_2 + A_{33}z_3$$

$$D_+\|z\| = (\beta(1 - n_p)(1 + 2a(N - I - R))v$$
$$+ \beta(1 - n_p)\left((N - I - R) + a(N - I - R)^2\right)\frac{v}{I}$$
$$+ \mu\frac{A}{I})z_1 + (\beta(1 - n_p)(1 + 2a(N - I - R))\, v)z_3$$
$$+ (\beta(1 - n_p)\left((N - I - R) + a(N - I - R)^2\right)\frac{v}{I})z_5$$
$$+ \alpha z_1 + \gamma z_2$$
$$+ (-\mu\frac{A}{I} - \beta(1 - n_p)(N - I - R)(1 + a(N - I - R))\frac{v}{I}$$
$$+ (a - \delta + \alpha + \gamma))z_3$$
$$\leq (-\beta(1 - n_p)(1 + 2a(N - I - R))v$$
$$- \beta(1 - n_p)\left((N - I - R) + a(N - I - R)^2\right)\frac{v}{I}$$
$$- \mu\frac{A}{I})|z_1| + \alpha|z_1| + \gamma|z_2|$$
$$+ (-\beta(1 - n_p)(1 + 2a(N - I - R))\, v)|z_3|$$
$$+ (\beta(1 - n_p)\left((N - I - R) + a(N - I - R)^2\right)\frac{v}{I})|z_5|$$
$$+ (-\beta(1 - n_p)(N - I - R)(1 + a(N - I - R))\frac{v}{I}$$
$$- \mu\frac{A}{I} + (a - \delta + \alpha + \gamma))|z_3|$$
$$\leq (-\beta(1 - n_p)(1 + 2a(N - I - R))v$$
$$- \beta(1 - n_p)\left((N - I - R) + a(N - I - R)^2\right)\frac{v}{I}$$
$$- \mu\frac{A}{I} + \alpha + \gamma)|z_1| + (\beta(1 - n_p)\left((N - I - R)\right.$$
$$+ a(N - I - R)^2\left.\right)\frac{v}{I})|z_5|$$
$$+ (-\beta(1 - n_p)(1 + 2a(N - I - R))\, v$$
$$- \beta(1 - n_p)(N - I - R)(1 + a(N - I - R))\frac{v}{I}$$
$$- \mu\frac{A}{I} + (a - \delta + \alpha + \gamma))|z_3|$$

Since $|z_5| < U_2(z) < |z_1| + |z_3|$, we obtain

$$D_+\|z\| \leq (-\beta(1 - n_p)(1 + 2a(N - I - R))v - \mu\frac{A}{I} - \delta + a + \alpha + \gamma)\|z\|.$$
$$\tag{5.6}$$

Subcase 2.2: $|z_1| < |z_2|$.

Then $\|z\| = |z_2| + |z_3| = z_2 + z_3$ and $U_2(z) < |z_2| + |z_3|$. On taking the right hand derivative of $\|z\|$, we obtain

$$D_+\|z\| = z_2' + z_3' = A_{22}z_2 + A_{23}z_3 + A_{26}z_6 + \alpha z_1 + \gamma z_2 + A_{33}z_3$$

$$D_+\|z\| = (-\beta(1 - n_p)(1 + 2a(N - I - R))v$$
$$- \beta(1 - n_p)\left((N - I - R) + a(N - I - R)^2\right)\frac{v}{I}$$
$$- \mu\frac{A}{I})z_2 + (-\beta(1 - n_p)(1 + 2a(N - I - R))\, v)z_3$$
$$+ (-\beta(1 - n_p)\left((N - I - R) + a(N - I - R)^2\right)\frac{v}{I})z_6$$
$$+ \alpha z_1 + \gamma z_2$$
$$+ (-\mu\frac{A}{I} - \beta(1 - n_p)(N - I - R)(1 + a(N - I - R))\frac{v}{I}$$
$$+ (a - \delta + \alpha + \gamma))z_3$$
$$= \alpha z_1 + (-\beta(1 - n_p)(1 + 2a(N - I - R))v$$
$$- \beta(1 - n_p)\left((N - I - R) + a(N - I - R)^2\right)\frac{v}{I}$$
$$- \mu\frac{A}{I} + \gamma)z_2 + (-\beta(1 - n_p)\left((N - I - R)\right.$$
$$+ a(N - I - R)^2\left.\right)\frac{v}{I})z_6$$
$$+ (-\beta(1 - n_p)(1 + 2a(N - I - R))\, v$$
$$- \beta(1 - n_p)(N - I - R)(1 + a(N - I - R))\frac{v}{I}$$
$$- \mu\frac{A}{I} + (a - \delta + \alpha + \gamma))z_3$$
$$\leq (-\beta(1 - n_p)(1 + 2a(N - I - R))v$$
$$- \beta(1 - n_p)\left((N - I - R) + a(N - I - R)^2\right)\frac{v}{I}$$
$$- \mu\frac{A}{I} + \gamma + \alpha)|z_2| + (\beta(1 - n_p)\left((N - I - R)\right.$$
$$+ a(N - I - R)^2\left.\right)\frac{v}{I})|z_6|$$
$$+ (-\beta(1 - n_p)(1 + 2a(N - I - R))\, v$$
$$- \beta(1 - n_p)(N - I - R)(1 + a(N - I - R))\frac{v}{I}$$
$$- \mu\frac{A}{I} + (a - \delta + \gamma + \alpha))|z_3|$$

Since $|z_6| < U_2(z) < |z_2| + |z_3|$, we obtain

$$D_+\|z\| \leq (-\beta(1 - n_p)(1 + 2a(N - I - R))v - \mu\frac{A}{I} - \delta + \alpha + a + \gamma)\|z\|.$$
$$\tag{5.7}$$

The remaining other 12 cases may also be dealt with in a similar way, we omit details. Combining the results of the four case presented here in Eqs. (5.4), (5.5), (5.6) and (5.7) as well as the remaining 12 cases, we get

$$D_+\|z\| \leq \max\{-\mu\frac{A}{I}, (-\beta(1 - n_p)(1 + 2a(N - I - R))v$$
$$- \mu\frac{A}{I} - \delta + a + \alpha + \gamma\}\|z\|.$$
$$\tag{5.8}$$

Hence, $\bar{\mu}(\mathcal{B}) \leq \max\{-\mu\frac{A}{I}, (-\beta(1 - n_p)(1 + 2a(N - I - R))v - \mu\frac{A}{I} - \delta + a + \alpha + \gamma)\}$. Therefore, $\bar{\mu}(\mathcal{B}) < 0$ on $int(\Omega)$, hence we can obtain that the endemic equilibrium E_2 is globally asymptotically stable. $\square$

6. Numerical simulation

To check the feasibility and effectiveness of the model proposed in (2.1), we perform numerical simulation. Different values of the parameters involved have been estimated with the help of

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Table 1
Estimated/assumed value of parameters.

Parameter	Value	Reference
A	40000	Assumed
μ	0.0002	Estimated
β	2.00×10^{-17}	Estimated
n_p	0.57	Estimated
a	0.10	Estimated
δ	0.00002	Estimated
α	0.00197	Estimated
c_1	0.9	Estimated
c_2	1.0	Estimated
c_3	0.1	Estimated
γ	0.0007	Estimated

best fit least squares approximation and the reproduction number has been calculated in each case. The estimated values of different parameters of the proposed model are given by Table 1.

In the model (2.1), the parameters a, β and n_p play a vital role in determining the effectiveness of the lockdown in the spread of the virus. As such, we consider a few cases, where by varying only these parameters and keeping the other parameters fixed, the value of the reproduction number R_0 becomes less than 1, that is, the spread of the disease decreases. Precisely, we fix parameters A, δ, α, γ, c_1, c_2, c_3, μ and vary a, β and n_p as per the table Table 2. It turns out that

1. In **Case (1)**, the reproduction number is 3.34989, which means that the spread of disease is endemic and that is happening in the present situation.
2. In **Case (2)**, n_p is increased, that is, when the susceptible class becomes aware and takes some precautions by wearing protective kits, washing hands or increasing immunity and the values of other parameters are kept fixed as in **Case (1)**, then the reproduction number decreases, but spread does not break although it slows down.
3. In **Case (3)**, a is decreased, that is, the rate at which the infected individuals move together with susceptible individuals without maintaining social distancing or taking any preventive measures, is decreased and values of other parameters are same as in **Case (2)**, then reproduction number decreases and becomes < 1, stating that the spread of virus is decreased.
4. Further in **Case (4)**, when β, that is, the transmission rate of the virus is increased and values of other parameters are kept same as in **Case (2)**, then the reproduction number decreases considerably and spread of virus deceases.
5. In the **Case (5)**, if the value of a is further decreased and values of other parameters remain same as in **Case (1)**, then the reproduction number decreases and becomes less than 1, which implies that the spread of the virus gets downward trend and the infection gets reduced.

7. Comparative analysis with real data

In this section, a comparative study of values of the variables obtained from the model (2.1) by suitable choice of parameters with the real data has been achieved with the help of figures. The data of the daily new COVID-19 cases in India has been compared with that obtained by the model (2.1) with proper choice of the parameters involved.

The real data regarding the actual number of COVID-19 cases in India have been collected from the official website of ICMR [9] and World Health Organization (WHO) [26]. Here we have considered the data of 68 days with effect from (w.e.f.) 25th March,
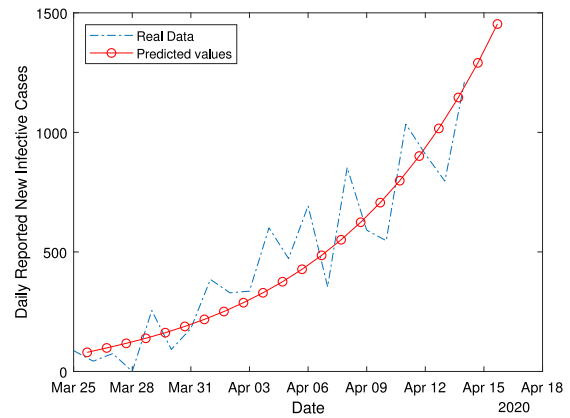

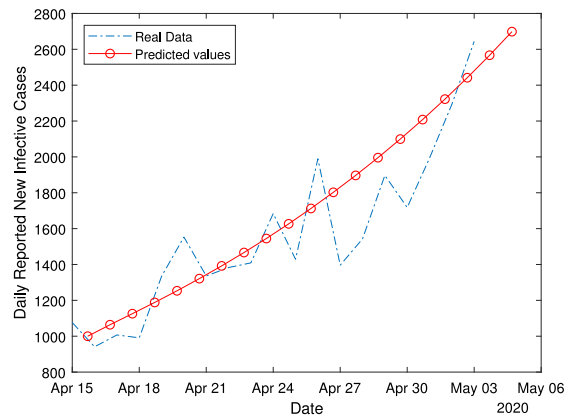**Fig. 1.** Nationwide Lockdown: Phase 1: 25 March 2020–14 April 2020 (21 days).

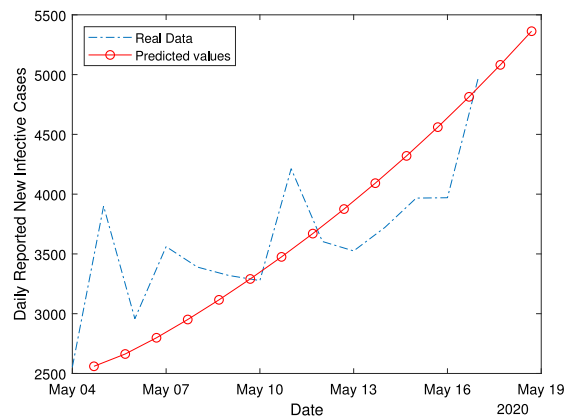**Fig. 2.** Nationwide Lockdown: Phase 2: 15 April 2020 – 3 May 2020 (19 days).

Fig. 3. Nationwide Lockdown: Phase 3: 4 May 2020–17 May 2020 (14 days).

2020 to 31st May, 2020 and divide it into 4 phases. Phase-1 w.e.f. 25th March, 2020 to 14th April, 2020 (21 days), Phase 2 w.e.f. 15th April, 2020 to 3rd May, 2020 (19 days), Phase 3 w.e.f. 4th May, 2020 to 17th May, 2020 (14 days) and Phase 4 w.e.f. 18th May, 2020 to 31st May, 2020 (14 days).

During Phase 1, it was a strict lock down and only persons in essential services were allowed to move, due to which, the number of daily reported infected cases were quite low. But to prevent the downfall of the economy it was decided by the government of India to relax the lock down in phases. In Fig. 1, we have plotted by blue and red lines the number of daily reported

Table 2

Reproduction number R_0 in cases when n_p, a and β are varied and other parameters are kept fixed.

Parameter	Case (1)	Case (2)	Case (3)	Case (4)	Case (5)
n_p	0.57	**0.80**	**0.80**	**0.80**	0.57
a	0.1	0.1	**0.06**	0.1	**0.02**
β	2.00×10^{-17}	2.00×10^{-17}	2.00×10^{-17}	**1.00×10^{-17}**	2.00×10^{-17}
$\mathcal{R}_0$	3.34989	1.55809	0.82715	0.779044	0.606435
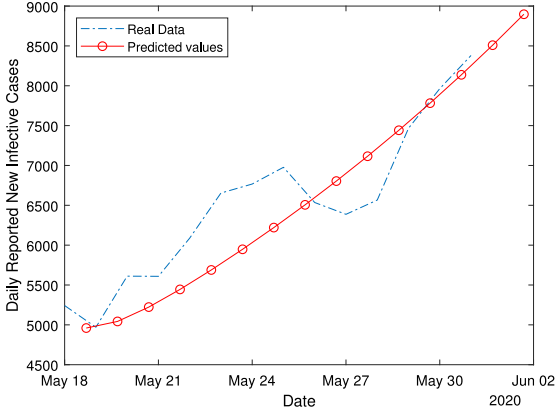
Fig. 4. Nationwide Lockdown: Phase 4: 18 May 2020–31 May 2020 (14 days).
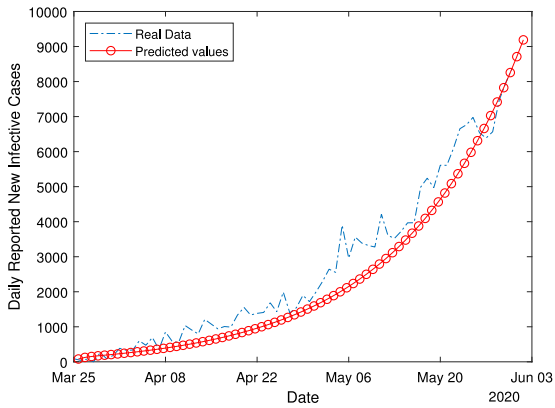
Fig. 5. Nationwide Lockdown: 25 March 2020–31 May 2020 (68 days).

Fig. 6. Variation of Reproduction number with respect to a.

Fig. 7. Variation of Reproduction number with respect to n_p.

Fig. 8. Variation of Reproduction number with respect to β.

infected actual cases as given by data available and that predicted with the help of the model (2.1) respectively, during Phase 1. It can be seen that the approximate value of the predicted infected cases is quite close to that of actual ones.

The lock down was further relaxed by allowing more and more activities like opening up of offices, shops etc. This allowed the more public to move freely which increased the chances of spread of the virus. Figs. 2–4 reflect situation of the actual and predicted infected cases during the Phase 2 (15th April, 2020 to 3rd May, 2020), Phase 3 (4th May, 2020 to 17th May, 2020) and Phase 4(18th May, 2020 to 31st May, 2020) respectively. As is seen from the figures the number of infected cases get increased in much higher proportion than that in earlier Phases and an approximate number of infected cases as calculated by the model (2.1) is quite near to the actual number of cases.

A cumulative affect of the nation wide lock down (complete as well as partial) w.e.f. 25th March, 2020 to 31st May, 2020 (68 days) has been presented in Fig. 5, showing that the error
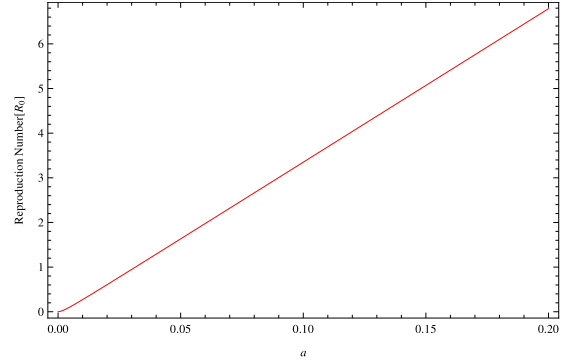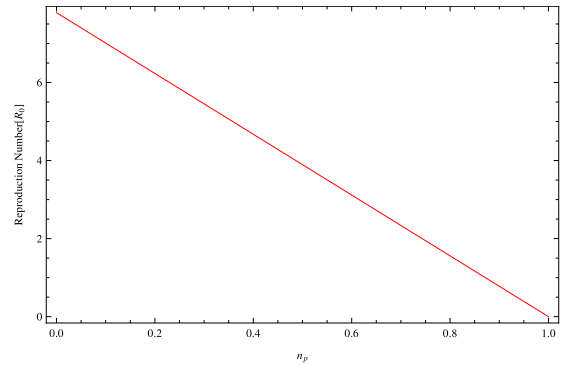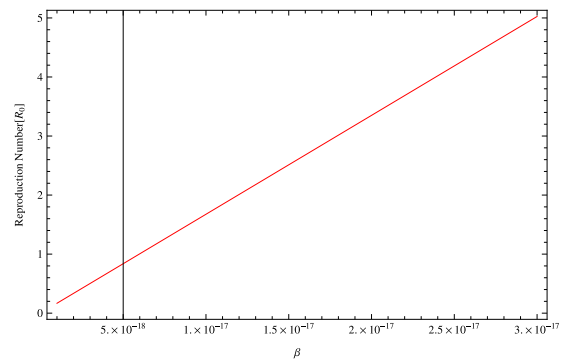
in predictions of the number of infected cases by model (2.1) to that of actual cases during the lock down period is comparable.

8. Conclusion

To predict the effectiveness of the lock down, we have plotted above the reproduction number against the variation of the parameters a and n_p. Fig. 6 shows that as the value of a is
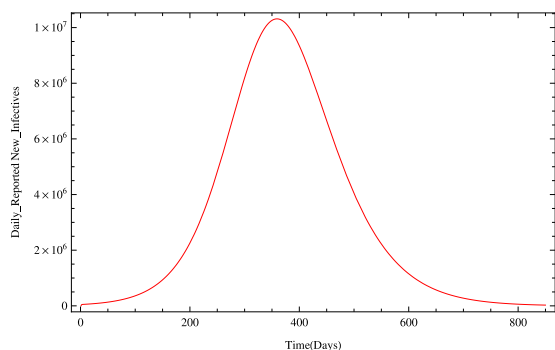
Fig. 9. Predicted Number of infected cases due to COVID-19 in India.

increased, that is, as the lock down is relaxed the reproduction number increases which implies that the spread of the virus is pandemic. Fig. 7 shows that with the increase in n_p, that is, if the self immunity is high and protective measures like social distancing, wearing masks, washing hands etc. are followed, then the reproduction number decreases, that is, the spread of virus decreases. Also, on varying β, that is, if the rate of transmission is increased the reproduction number increases, indicating the increase in the spread of virus as shown in Fig. 8.

Assuming the situation that the lock down be relaxed further, that is, more and more susceptible population move out of their residence freely. The possibility of these people coming in contact with the symptomatic or asymptomatic infected persons increases and hence getting infected with COVID-19 virus also increases. With the suitable choice of parameters, using the model (2.1), it is shown in Fig. 9 that the spread of COVID-19 virus will come to an end in India by itself but may take time approximately 2 years. As per model, the peak value of the total number of infected population in India is approximately 1×10^7 and is expected to reach by the end of January, 2021. If some medication of a vaccine is developed in the mean time, then the spreading of COVID-19 may end up early.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] National health commission of the people's Republic of China. 2020, Feb 14, http://www.nhc.gov.cn/xcs/yqtb/202002/553ff43ca29d4fe88f3837d49d6b6ef1.shtml.

[2] Rodriguez KM, Chowell G, Cheung CH, et al. Epidemic doubling time of the 2019 novel corona virus outbreak by province in mainland China. 2020, medRxiv https://www.medrxiv.org/content/early/2020/02/07/2020.02.05.20020750.full.pdf.

[3] Yang Y, Lu QB, Liu MJ, et al. Epidemiological and clinical features of the 2019 novel corona virus outbreak in China. 2020, medRxiv https://www.medrxiv.org/content/early/2020/02/11/2020.02.10.20021675.full.pdf.

[4] Peng L, Yang W, Zhang D, Zhuge C, Hong L. Epidemic analysis of COVID-19 in China by dynamical modeling. 2020, 25 Jun, arXiv:2002.06563v2.

[5] Sardar T, Nadim SS, Chattopadhyay J. Assessment of 21 days lock down effect in some states and overall India: A predictive mathematical study on COVID-19 outbreak. 2020, 7 April, arXiv:2004.03487v1.

[6] Sardar T, Rana S. Effective lock down and role of hospital-based COVID-19 transmission in some Indian states: An outbreak risk analysis. 2020, 3 May, arXiv:submit/3160351.

[7] Why are hospitals hotbeds of COVID-19 transmission? 2020, Published on: 2020-04-20, https://www.thehindu.com/sci-tech/why-are-hospitals-hotbeds-of-covid-19-transmission/article31386208.ece.

[8] Zhao S, Lin Q, Ran J, et al. Preliminary estimation of the basic reproduction number of novel corona virus (2019-nCoV) in China, from 2019 to 2020: A data-driven analysis in the early phase of the outbreak. Int J Infect Dis 2020;92:214–7. http://dx.doi.org/10.1016/j.ijid.2020.01.050.

[9] Indian Council of Medical research. 2020, https://www.icmr.nic.in/.

[10] Nowak M, May RM. Virus dynamics. Oxford University Press; 2000.

[11] Rai RK, Misra AK, Takeuchi Y. Modeling the impact of sanitation and awareness on the spread of infectious diseases. Math Biosci Eng 2019;16(2):667–700.

[12] Heesterbeek JAP. Mathematical epidemiology of infectious diseases, model building, analysis and interpretation, vol. 5. Hoboken: Wiley; 2000.

[13] Hethcote HW. Three basic epidemiological models. Biomathematics 1989;119–44.

[14] Driessche PVD, Watmough J. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Math Biosci 2002;180:29–48.

[15] Perasso A. An introduction to the basic reproduction number in mathematical epidemiology. ESAIM: Proc Surv 2018;62:123–38.

[16] Hale JK. Ordinary differential equations. New York: Wiley-Interscience; 1969.

[17] Thieme HR. Persistence under relaxed point-dissipativity (with application to an endemic model). SIAM J Math Anal 1993;24:407–35.

[18] Freedman H, Ruan S, Tang M. Uniform persistence and flows near a closed positively invariant set. J Dynam Differential Equations 1994;6(4):583–600.

[19] Samanta S, Rana S, Sharma A, Misra AK, Chattopadhyay J. Effects of awareness programs by media on the epidemic outbreaks; A mathematical model. Appl Math Comput 2013;219(12):6965–77.

[20] Verma H, Pandya N, Mishra VN, Mathur P. Modeling and analysis of media's role in popularizing the blue whale game. Int J Math Ind 2020. http://dx.doi.org/10.1142/S2661335220500082.

[21] Li MY, Moldowney JS. A geometric approach to global-stability problem. SIAM J Math Anal 1996;27:1070–83.

[22] Smith RA. Some applications of Hausdorff dimension inequalities for ordinary differential equations. Proc Roy Soc Edinburgh Sect A 1986;104:235–59.

[23] Li MY, Moldowney J. On R.A. Smith's autonomous convergence theorem. Rocky Mountain J Math 1995;25(1):365–78.

[24] Zhou XY, Cui JA, Zhang ZH. Global results for a cholera model with imperfect vaccination. J Franklin Inst 2012;349:770–91.

[25] Sharma S, Kumari N. Dynamics of a waterborne pathogen model under the influence of environmental pollution. Appl Math Comput 2019;346:219–43.

[26] Who. Corona virus disease (COVID-19) outbreak. 2020, https://www.who.int/emergencies/diseases/novelcoronavirus-2019.